The shape and stability of a bubble at the axis of a rotating liquid

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The shape of a bubble of one liquid inside a denser body of liquid which rotate rigidly together is determined, the effect of gravity being neglected. When the angular velocity of the liquids is zero the bubble assumes a spherical form, and with increasing angular velocity the bubble flattens at the equator and the length increases. It is found that the length of the bubble is asymptotically proportional to the four-thirds power of the angular velocity. If the speed of the rotation is held constant and the volume is increased, then the bubble elongates, the radius approaches a limiting value, and the bubble length increases almost linearly with the volume. This result suggests a method whereby the interfacial surface tension can be measured.

In the second part of the paper the stability of a long bubble subjected to small amplitude axisymmetric disturbances sinusoidal in the axial direction is investigated. The relation between the wave-number and angular velocity for neutral stability is elliptic. When account is taken of the decrease in the radius of the undisturbed bubble with increase in the angular velocity, it is found that the bubble is stable to all wave-lengths provided the radius attains at least 63 % of the limiting value. A criterion is then found for the minimum length of the bubble consistent with stability.

1. The shape of a bubble in equilibrium

The equilibrium of a revolving isolated finite mass of liquid under the action of capillary force was discussed by Lord Rayleigh (1914). He found a solution in which the bubble is a surface of revolution which meets the axis of the rotation. If there is no rotation then the bubble assumes a spherical form while under the influence of rotation, the sphere flattens at the poles and the oblateness increases with the angular velocity.

In this paper account is taken of a body of liquid surrounding the bubble. The shape of this bubble of one liquid (or fluid) which is immersed in and rotates with another liquid is determined by the balance of the surface tension and the hydrodynamic pressure. If the effect of the gravitational field is negligible and if the external body of liquid is denser than the bubble, then the interface is a surface of revolution which meets the axis of rotation. This suggests the use of cylindrical polar co-ordinates (r, θ, z) with the z-axis as the axis of the rotation.

The pressure distributions in the two media are given by

$$\left.\begin{aligned} p_1 &= \tfrac{1}{2}\rho_1\omega^2r^2+p_{01} \\ \text{and} \qquad p_2 &= \tfrac{1}{2}\rho_2\omega^2r^2+p_{02}, \end{aligned}\right\} \qquad (1)$$

where ω is the angular velocity of the system and the suffices 1 and 2 correspond to the bubble and the liquid outside the bubble, respectively.

The problem is to find the equation of the interface, $r = f(z)$ say, such that the pressure discontinuity at every point of this surface is TJ, where T is the surface tension and J is the total curvature. The kind of solution sought is one for which the plane section through the axis of the rotation has no points of inflexion. Other shapes are mathematically possible but they do not correspond to observation.

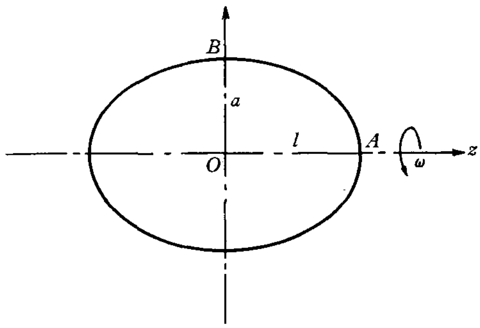

FIGURE 1. A section by a plane through the axis of Oz.

For a surface of revolution

$$J = \frac{1}{f}\frac{d}{df}\left(\frac{f}{(1+f'^2)^{\frac{1}{2}}}\right); \tag{2}$$

hence, the differential equation governing the shape of the bubble is

$$\frac{d}{df}\left(\frac{f}{(1+f'^2)^{\frac{1}{2}}}\right) = \frac{(p_{01}-p_{02})f}{T} - \frac{(\rho_2-\rho_1)\,\omega^2 f^3}{2T}, \tag{3}$$

with $f = 0$ at $z = \pm l$ and $f' = 0$ at $z = 0$.

This equation can be integrated to give

$$(1+f'^2)^{-\frac{1}{2}} = \frac{(p_{01}-p_{02})f}{2T} - \frac{(\rho_2-\rho_1)\,\omega^2 f^3}{8T}; \tag{4}$$

the constant of integration is zero, because the bubble meets the axis of the rotation. This shows that the interface cuts the z-axis at right angles.

If the maximum radius of the bubble $OB = a$ and the curvature at the pole of the section through the axis of the bubble is $-(1+e)/a$, then

$$\frac{1+e}{a} = \left\{\frac{d}{df}(1+f'^2)^{-\frac{1}{2}}\right\}_{f=0} = \frac{p_{01}-p_{02}}{2T}. \tag{5}$$

Using the condition that $f' = 0$ at $f = a$ and combining (4) and (5) leads to

$$\frac{e}{a^3} = \frac{(\rho_2-\rho_1)\,\omega^2}{8T}. \tag{6}$$

The differential equation now simplifies to the form

$$(1+f'^2)^{-\frac{1}{2}} = (1+e)\frac{f}{a} - e\frac{f^3}{a^3}. \tag{7}$$

Since the curvature of the section through the axis of the bubble is

$$\frac{3ef^2}{a^3} - \frac{1+e}{a} \tag{8}$$

and remains negative and $(\rho_2 - \rho_1) > 0$, it follows that

$$0 \leqslant e < \tfrac{1}{2}. \tag{9}$$

This inequality is equivalent to

$$\left.\begin{aligned} a^3\omega^2(\rho_2-\rho_1) &< 4T \\ \text{and} \qquad 2T \leqslant a(p_{01}-p_{02}) &< 3T, \end{aligned}\right\} \tag{10}$$

which shows that the maximum radius of the bubble adjusts itself to a value for which the capillary force can balance the combined effects of the centripetal force and the pressure difference on the axis of the rotation. The second inequality in (10) also shows that the pressure difference at the pole of the bubble must be greater than $2T/a$, the pressure difference required to maintain a spherical bubble of radius a. This difference is not a controllable parameter but adjusts itself once the angular velocity and the volume of the bubble is given. This adjustment is possible because of the finite compressibility of real liquids.

From (7) the length and volume of the bubble can be found; namely

$$l = a\int_0^1 \frac{A}{(1-A^2)^{\frac{1}{2}}}\,dx, \quad V = 2\pi a^3 \int_0^1 \frac{x^2 A}{(1-A^2)^{\frac{1}{2}}}\,dx, \tag{11}$$

where $A = x(1+e-ex^2)$.

Using the identity

$$\frac{d}{dx}\left(\frac{2\pi a^3}{3e}(1-A^2)^{\frac{1}{2}}\right) = \frac{2\pi a^3 x^2 A}{(1-A^2)^{\frac{1}{2}}} - \frac{2\pi a^3(1+e)A}{3e(1-A^2)^{\frac{1}{2}}} \tag{12}$$

and integrating over the range (0, 1) leads to

$$V = \frac{2\pi a^3}{3e}\left\{\frac{(1+e)\,l}{a} - 1\right\}. \tag{13}$$

Equations (6), (11) and (13) shows that, as the angular velocity increases from zero, the parameter e increases within the range $0 \leqslant e < 0{\cdot}5$. Moreover, the limiting value of e must be $0{\cdot}5$; for, if it is less, then the integrals in (11) converge which shows that the angular velocity is a bounded function of the volume, surface tension and the density difference. This leads to a contradiction.

We can now deduce that the bubble is quasi-cylindrical when the angular velocity is large. This depends on the behaviour of the integrals in (11) when the parameter e is near $0{\cdot}5$. There is then a 'tendency to diverge', i.e. a small increase in e produces a large increment in the ratio l/a. It follows from (13) that the volume is asymptotically proportional to $2\pi la^2$, the volume of a cylinder having the same length and radius as the bubble.

Calculations of the bubble dimensions

The equation determining the length of the bubble can be transformed to a more convenient form by the substitution

$$\left.\begin{aligned} x^2 &= 1 - L\cot^2\theta, \\ \text{where}\qquad L &= e^{-1} - \tfrac{1}{2} + (e^{-1} + \tfrac{1}{4})^{\frac{1}{2}}. \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} \text{Then,}\qquad l &= a\int_{\alpha}^{\frac{1}{2}\pi} \frac{(1 + eL\cot^2\theta)\,d\theta}{eL^{\frac{1}{2}}(l - k^2\sin^2\theta)^{\frac{1}{2}}} \\ \text{with}\qquad k^2 &= [2(L+1)^{\frac{1}{2}} - 1]/L \quad\text{and}\quad \alpha = \tan^{-1}L^{\frac{1}{2}}. \end{aligned}\right\} \tag{15}$$

This integral can be written in terms of elliptic integrals of the first and second kind:

$$l/a = e^{-1}\cot\alpha[F(k, \tfrac{1}{2}\pi) - F(k, \alpha)] - \tan\alpha[E(k, \tfrac{1}{2}\pi) - E(k, \alpha)] + 1 - \cos\alpha. \tag{16}$$

The latter form has been used to compute the bubble dimensions in terms of the independent variable $\sin^{-1}k$, which takes all the values in the range $(0, \frac{1}{2}\pi)$ when $0 \leqslant e < 0{\cdot}5$. The appropriate formulae for the determination of the physical constants are

$$\left.\begin{aligned} (L+1)^{\frac{1}{2}} &= \frac{1 + (1 - k^2 + k^4)^{\frac{1}{2}}}{k^2}, \\ e^{-1} &= L + 1 - (L+1)^{\frac{1}{2}} \\ \text{and}\qquad \frac{eV}{a^3} &= \frac{2\pi}{3}\left((1+e)\frac{l}{a} - 1\right). \end{aligned}\right\} \tag{17}$$

Table of computed data

Values of V/c^3, a/c and l/a $[c^3 = 8T/(\rho_2 - \rho_1)\,\omega^2]$ tabulated at intervals of $\sin^{-1}k$.

$\sin^{-1}k$	V/c^3	a/c	l/a	$\sin^{-1}k$	V/c^3	a/c	l/a
15°	0·005	0·106	1·001	65°	2·078	0·683	1·511
20°	0·016	0·157	1·004	70°	2·842	0·722	1·712
25°	0·040	0·212	0·010	75°	3·863	0·753	1·993
30°	0·085	0·271	0·020	77°	4·378	0·763	2·139
35°	0·161	0·333	1·038	79°	4·982	0·772	2·315
40°	0·280	0·397	1·067	81°	5·709	0·779	2·530
45°	0·457	0·461	1·109	83°	6·622	0·785	2·805
50°	0·705	0·523	1·169	85°	7·844	0·789	3·181
55°	1·045	0·582	1·252	87°	9·706	0·792	3·764
60°	1·547	0·636	1·363	89°	13·684	0·794	5·024

These calculations show that the radius of the cylindrical portion of the bubble is closely approximated by the formula $a^3 = 0{\cdot}5c^3$ when the angular velocity of the system is large. In fact, for $V/c^3 = 5$, 10 or 14 this approximation is within 3, 0·2 or 0·02 %, respectively.

The equations of the asymptotes to the three graphs of figure 2 are

$$\left.\begin{aligned} V/c^3 &= \pi(l/a - \tfrac{2}{3}), \\ V/c^3 &= 2^{\frac{1}{3}}\pi(l/c - 2^{\frac{2}{3}}/3), \\ \text{and}\qquad a^3 &= 0{\cdot}5c^3. \end{aligned}\right\} \tag{18}$$

These serve as a check on the calculations as well as an indication of the shape of the bubble when V/c^3 is large.

For small values of V/c^3 the bubble is an ellipsoid correct to the first order in e. This follows from the inequality

$$(1-e)^{-1} < l/a < (1-2e)^{-1}(1-e), \tag{19}$$

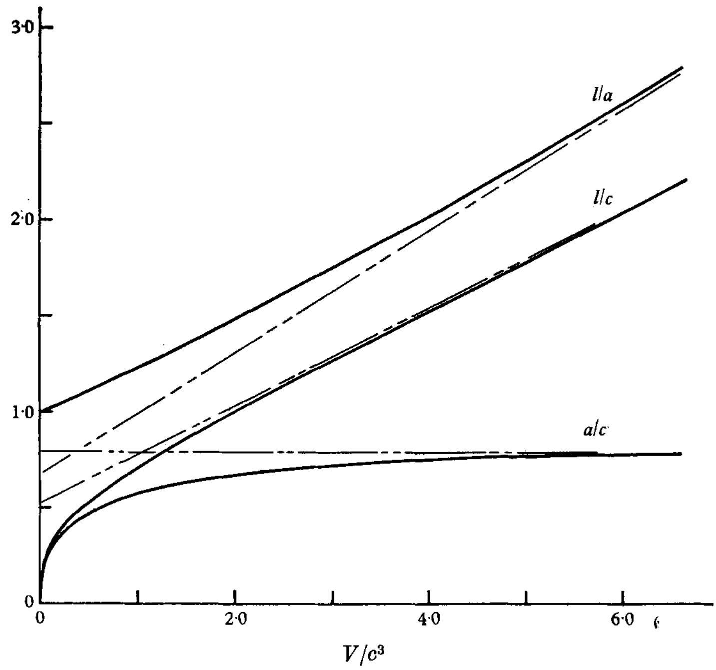

FIGURE 2. The change in bubble dimensions with angular velocity. The broken lines are asymptotes.

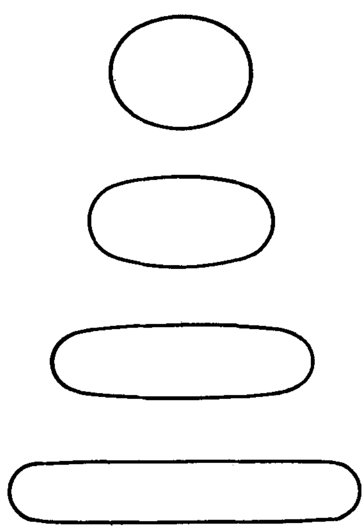

FIGURE 3. The bubble dimensions at angular velocities of 1, 2, 3 and 4.

which expresses the condition that the volume of the bubble lies between that of an enclosed prolate spheroid with the same major axes and that of a domed cylinder with hemispherical ends.

To show that the bubble does enclose such a spheroid it is sufficient to consider the difference between the abscissae of the plane section through the bubble and the ellipse having the same major and minor radii. This difference is given by

$$\left.\begin{aligned} G(f) &= a\int_0^{(1-f^2/a^2)^{\frac{1}{2}}}\left(F\{(1-x^2)^{\frac{1}{2}}\}-\int_0^1 F\{(1-y^2)^{\frac{1}{2}}\}\,dy\right)dx,\\ \text{where}\qquad F(x) &= \frac{1+e-ex^2}{[(1-ex^2)^2-e^2x^2]^{\frac{1}{2}}} \end{aligned}\right\} \tag{20}$$

and can be shown to be positive if $0 < r < a$ and $0 < e < 0{\cdot}5$.

If the volume of the bubble and the density difference between the two liquids is known, then a measurement of length and angular velocity is sufficient to determine the interfacial surface tension. If the volume of the bubble is so large that the bubble had nearly attained its limiting radius then the second equation of (18) can be used to calculate T. The degree of accuracy of this method can be estimated from further measurements.

An investigation of the cubic equation from which T is to be calculated shows that the larger positive root is appropriate; for otherwise l^3/c^3 is not greater than $0{\cdot}5$ and the equations (18) are not then applicable.

2. The stability of a long bubble to axisymmetric disturbances

Some conditions of stability of an almost cylindrical bubble can be inferred from a paper by Lord Rayleigh (1892). He showed that in a non-rotating field the bubble was unstable to axisymmetric disturbances whose wave-length exceeded the circumference of the bubble, because then the surface area decreases with a consequent release of surface energy. This surplus energy enables the disturbances to grow.

Hocking (1960) went a little further by considering the stability of a rigidly rotating column of liquid surrounded by a fluid of negligible density. Here, the rotation has a destabilizing influence and the wave-length of a neutral axisymmetric disturbance is shorter than the circumference of the column.

The question of stability of a cylindrical bubble of finite volume surrounded by a denser fluid is to be investigated. The rotation has a stabilizing influence for a bubble of given radius, and for a given angular velocity the neutral disturbance has a wave-length greater than the circumference of the bubble. The decrease in bubble radius which accompanies an increase in the angular velocity suggests that a previously neutral disturbance of given wave-length may become unstable. One of the results to be established here is that the stabilizing influence of the increased centripetal force offsets the approach to instability, due to decrease of radius, and that a long bubble of sufficiently small radius is stable at all wave-lengths. It is here assumed that the interface is a cylinder and that end effects may be neglected.

We shall consider the effect of a disturbance of small amplitude on the bubble. The steady state solution of this problem is given by

$$\bar{u} = 0, \quad \bar{v} = \omega r, \quad \bar{w} = 0, \quad \bar{p} = \tfrac{1}{2}\rho\omega^2 r^2 + p_0,$$

so, on putting (u, v, w) and p as the departures from the steady state solution, the equations of motion become

$$\frac{\partial u}{\partial t} - 2\omega v = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \quad \frac{\partial v}{\partial t} + 2\omega u = 0, \quad \frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{21}$$

and the continuity equation becomes

$$\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{\partial w}{\partial z} = 0. \tag{22}$$

These equations have been linearized by neglecting the squares of the perturbed components of velocity.

Equation (22) shows that there exists a stream function $\psi = \psi(r, z, t)$ such that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{23}$$

This can be used to eliminate p from the equations of motion; hence

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right) + 4\omega^2 \frac{\partial^2 \psi}{\partial z^2} = 0. \tag{24}$$

To find the solution of this differential equation we can make a Fourier analysis with respect to z and assume a exponential dependence on the time. This is suggested by the theory of normal modes.

Let

$$\psi = \exp i(\sigma t + \mu z) f(r); \tag{25}$$

then substitution in (24) yields the Bessel-type differential equation

$$\left.\begin{aligned} r^2 f'' - r f' + n^2 r^2 f &= 0, \\ \text{with} \qquad n^2 &= \mu^2\left(\frac{4\omega^2}{\sigma^2} - 1\right). \end{aligned}\right\} \tag{26}$$

If the equation of the disturbed interface is taken to be

$$r = a + \eta + O(\eta^2), \tag{27}$$

where $\eta = \eta(z, t)$ measures the displacement of the surface from its undisturbed position, then, since the interface moves with the liquid,

$$u = \frac{\partial \eta}{\partial t} \quad \text{at} \quad r = a. \tag{28}$$

The above equations lead to

$$\left.\begin{aligned} p_{1,2} &= \frac{n\sigma\rho}{\mu} \exp i(\sigma t + \mu z)\,[\alpha_{1,2} J_0(nr) - \beta_{1,2} Y_0(nr)] \\ \text{and} \qquad \eta &= \frac{\mu}{\sigma} \exp i(\sigma t + \mu z)\,\alpha_1 J_1(na), \end{aligned}\right\} \tag{29}$$

provided that $n \neq 0$. The constants of integration $\alpha_{1,2}$ and $\beta_{1,2}$ are to be determined from the boundary conditions.

The stability criterion

The boundary conditions that apply to this problem are

(a) that the pressure discontinuity at the interface be balanced by the surface tension;

(b) that the two liquids remain in contact throughout the disturbance;

(c) that the normal velocity at an outer cylindrical boundary at $r = b$ be zero.

Since the total curvature at a point on the interface is $(1-\eta/a)/a-\eta''$, it follows that the pressure discontinuity equation is

$$\frac{1}{4}\left\{\frac{T(\mu^2a^2-1)}{\omega^2a^3\rho_2}+\frac{\rho_2-\rho_1}{\rho_2}\right\} = \frac{na}{\mu^2a^2+n^2a^2}\frac{J_0(na)}{J_1(na)}\left\{\frac{\rho_1}{\rho_2}+\frac{Y_0(na)/J_0(na)-Y_1(nb)/J_1(nb)}{Y_1(nb)/J_1(nb)-Y_1(na)/J_1(na)}\right\}, \tag{30}$$

from which the stability criterion can be found.

The next step is to find a necessary and sufficient condition for a small amplitude axisymmetric disturbance sinusoidal in the axial direction to be stable. We shall consider the right-hand side of (30) and show that it is positive whenever $\sigma^2 > 0$ and that it tends to zero as $\sigma^2 \to 0$ in a suitable manner.

Suppose that $\sigma^2 > 4\omega^2$ (which implies stability); then the right-hand side of the above equation is

$$\frac{ma}{\mu^2a^2-m^2a^2}\frac{I_0(ma)}{I_1(ma)}\left\{\frac{\rho_1}{\rho_2}+\frac{K_0(ma)/I_0(ma)+K_1(mb)/I_1(mb)}{K_1(ma)/I_1(ma)-K_1(mb)/I_1(mb)}\right\}, \tag{31}$$

where $m^2 = \mu^2(1-4\omega^2/\sigma^2) > 0$. It follows that this is positive for all values of $m > 0$ and hence, that

$$\mu^2a^2+(\rho_2-\rho_1)\,a^3\omega^2/T > 1. \tag{32}$$

If $0 < \sigma^2 < 4\omega^2$, then the region outside the bubble may contain nodal surfaces whose positions are given by those roots of the equation

$$J_1(nb)\,Y_1(nr)-J_1(nr)\,Y_1(nb) = 0 \tag{33}$$

which satisfy $0 < r < b$.

For convenience b_1 will be defined as the distance from the axis of the rotation to the first nodal surface outside the bubble. Hence na and nb_1 lie between the same two zeros of $J_1(x)$. Since

$$Y_0(x)\,J_1(x)-Y_1(x)\,J_0(x) = 1/x, \tag{34}$$

the right-hand side of (30) can be written

$$\frac{na}{\mu^2a^2+n^2a^2}\left\{-\frac{(\rho_2-\rho_1)}{\rho_2}\frac{J_0(na)}{J_1(na)}+\frac{1/naJ_1^2(na)}{Y_1(nb_1)/J_1(nb_1)-Y_1(na)/J_1(na)}\right\} \tag{35}$$

with b replaced by b_1, because b_1 is a root of (33). This term is positive when $J_0(na)\,J_1(na) < 0$.

If $J_0(na)\,J_1(na) > 0$, then the right-hand side of (30) is clearly positive. This shows that, for any stable disturbance, condition (32) is necessary.

To show that this condition is also sufficient it is enough to find the neutral curve at the limit of instability. This is done by using the asymptotic formulae for modified Bessel functions and letting $\sigma^2 \to 0$ through other than real positive values.

Now $$\frac{I_0(ma)}{I_1(ma)} \quad\text{and}\quad \frac{K_0(ma)/I_0(ma)+K_1(mb)/I_1(mb)}{K_1(ma)/I_1(ma)-K_1(mb)/I_1(mb)} \to 1 \quad\text{as}\quad m\to\infty, \tag{36}$$

which shows that the right-hand side of (31) tends to zero. Hence

$$\mu^2a^2+(\rho_2-\rho_1)\,a^3\omega^2/T > 1 \tag{37}$$

is a necessary and sufficient condition for stability.

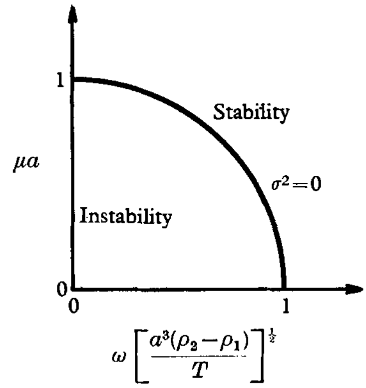

FIGURE 4. The neutral curve.

For an actual bubble it has been shown that the radius was dependent upon the physical constants of the system. In fact the limiting radius is given by

$$(\rho_2-\rho_1)\,\omega^2a_c^3 = 4T. \tag{38}$$

If the bubble is long and end effects are negligible then the stability criterion can be expressed in the simplified form

$$\mu^2a^2 > 1-4a^3/a_c^3, \tag{39}$$

which shows that the bubble is stable to all wave-lengths if its radius has attained at least 63 % of the limiting value. This is certainly true if the bubble length exceeds $l_c = 0{\cdot}54c$.

When the bubble radius is less than the critical value the centripetal force is insufficient to balance the destabilizing influence of the capillary force unless the wave-length of the disturbance is sufficiently small.

In conversation Dr G. K. Batchelor pointed out to the author that viscosity can have no effect on the curve of neutral stability because it corresponds to a steady rigid body rotation. Thus all the results obtained here concerning the condition for instability apply equally to a bubble of viscous liquid immersed in a second viscous liquid.

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